Appendix A

Gaunt factor evaluation

This appendix derives equations (4.7) and (4.9), for use in the calculation of the Gaunt factors for the new coupling scheme. Coupling of the angular momentum of a valence electron to the core configuration can be conveniently expressed with the introduction of 'Clebsch-Gordan' coefficients (sometimes referred to as 'vector coupling' coefficients and equivalently expressed as 'Wigner 3-j' coefficients). Consider the wavefunction $\Psi(j_1 j_2 j m)$ such that j is the resultant angular momentum due to j_1 and j_2, with corresponding magnetic quantum number m. The Clebsch-Gordan coefficient ($C^{j_1 j_2 j}_{m_1 m_2 m}$) can be used to uncouple $\Psi(j_1 j_2 j m)$ into the constituent wavefunctions for j_1 and j_2. $C^{j_1 j_2 j}_{m_1 m_2 m}$ and is defined such that

$$\Psi(j_1 j_2 j m) = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} \Phi(j_1 j_2 m_1 m_2) \tag{A.1}$$

where $\Phi(j_1 j_2 m_1 m_2) = \Psi(j_1 m_1)\Psi(j_2 m_2)$. Angular momentum problems reduce to a set of these Clebsch-Gordan coefficients which can be further reduced to a set of 'Wigner 6-j' and higher coefficients using various transformations and symmetry properties. There exists an elegant technique, first introduced by Levinson (1955) for the manipulation of angular momentum coefficients, whereby one converts the coefficients into graphical form and manipulates the resultant diagrams. The various mathematical transformations of the Clebsch-Gordan coefficients all have corresponding geometric transformations on the diagram. The graphical method is described in standard quantum angular momentum text books such as Rose (1986), Edmonds

(1974), with the most complete description being found in chapter 7 of Brink and Satchler (1968). For completeness, the key details are reproduced here in appendix A.1. The new Gaunt factors are then derived using these rules in appendix A.2.

A.1 Basic rules for graphical expression and manipulation of Clebsch-Gordan coefficients

The Clebsch-Gordan coefficients can be transformed into Wigner 3-j coefficients as shown in equation A.2.

$$C^{abc}_{\alpha\beta\gamma} = (-1)^{a-b-\gamma}(2c+1)^{1/2}\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \tag{A.2}$$

where $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ is the wigner 3-j coefficient. This is expressed graphically by associating a straight line with each of the angular momentum vectors a, b and c and joining the lines at a single point or 'node'. The Clebsch-Gordan symbol is drawn as in figure A.1 and the Wigner 6-j as in figure A.2. - note that each node has a sign attached to it. By convention the magnetic quantum number associated with each angular momentum vector is omitted from the diagram. Also note that some vectors have directional arrows attached to them.

The rules for graphical manipulation of Clebsch-Gordan coefficients are as follows:

1. Any geometric deformation that preserves the order of the arrows and signs on each node is allowed. If the cyclic order at any node is changed there must be a corresponding change in the sign of the node.

2. A change in the cyclic order of the 3-j symbol results in a change in the sign of the node as shown in figure A.3.

3. Two lines representing the same total angular momentum can be joined together as in figure A.4

4. A line with two oppositely directed arrows is equivalent to a line with no arrows

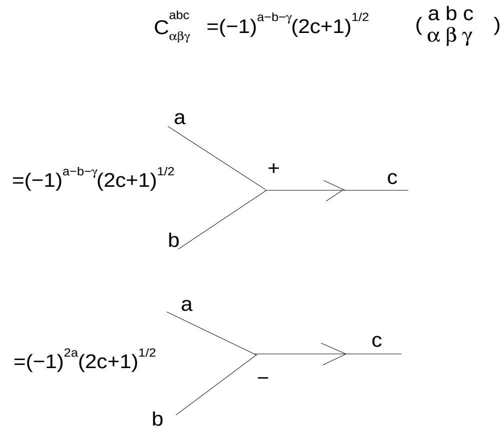


Figure A.1: Graphical representation of the Clebsch-Gordan coefficient

5. A line corresponding to an angular momentum a with two arrows in the same direction is equivalent to a line with no arrows times a factor $(-1)^{2a}$.

6. If an arrow on a line with angular momentum a is reversed, the graph must be multiplied by a factor $(-1)^{2a}$.

7. Three arrows may be added to a node, one to each line, without changing the value of the graph. The arrows must all be directed either into or out from the node.

The ‘Wigner-Eckart’ theorem can be used to reduce the matrix $< jm|T_{\lambda\mu}|j'm' >$, where $T_{\lambda\mu}$ is an irreducible tensor operator (such as a spherical harmonic), to extract the dependence on the magnetic quantum numbers. The theorem states that

$$< jm|T_{\lambda\mu}|j'm' >= \frac{< j||T_{\mu}||j' >}{\sqrt{2j+1}} C^{j'\lambda j}_{m'\mu m} \tag{A.3}$$

Note that the magnetic quantum number is now contained in a Clebsch-Gordan coefficient.

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \equiv \Sigma\,(-1)^{a+e+c-\alpha-\varepsilon-\gamma} \begin{pmatrix} a & f & c \\ \alpha & \phi & -\gamma \end{pmatrix} \begin{pmatrix} c & d & e \\ \gamma & \delta & -\varepsilon \end{pmatrix} \begin{pmatrix} e & b & a \\ \varepsilon & \beta & -\alpha \end{pmatrix} \begin{pmatrix} b & d & f \\ \beta & \delta & \phi \end{pmatrix}$$

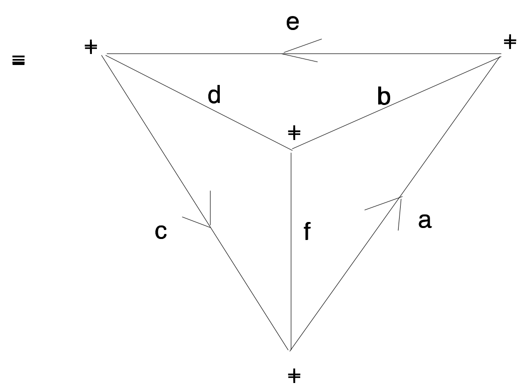


Figure A.2: Graphical representation of the Wigner 6-j coefficient. The summation is over all magnetic quantum numbers.

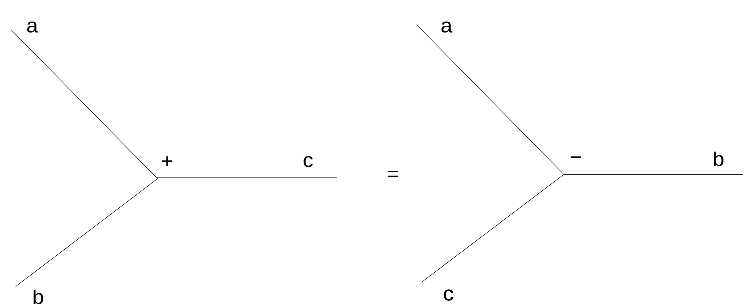


Figure A.3: Change in cyclic order of Wigner 3-j symbol

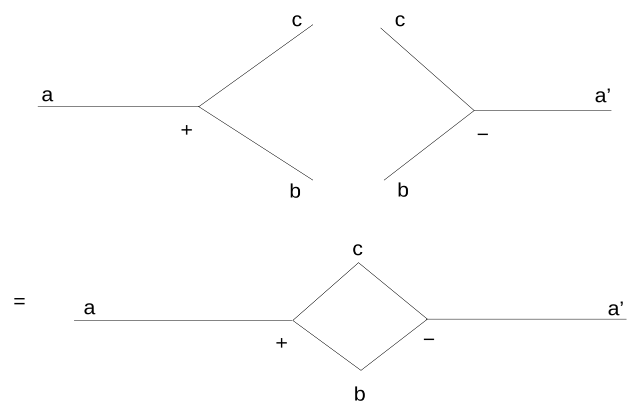


Figure A.4: Combining Wigner 3-j symbols

A.2 Application of the graphical manipulation method for Clebsch-Gordan coefficients

The evaluation of Gaunt factors, such as the bound-bound expression

$$g^I_{i,i'}(\nu,\nu') = \frac{\sqrt{3}}{\pi 2^4}\left(\frac{E-E'}{z_1^2 I_H}\right)^4 \frac{1}{\omega_\gamma} Q_{i,i'} R^I_{i,i'}(\nu,\nu') \tag{A.4}$$

reduces to a need to evaluate Q and R, where R is the radial integral component and Q contains the angular factors. In the case of a highly excited level built on a parent level, Q and R can be rewritten

$$Q_{i,i'} R^I_{i,i'}(\nu,\nu') = < (S_P L_P J_P M_P) nljmJM|\underline{r}|(S'_P L'_P J'_P M'_P) n'l'j'm'J'M' > \tag{A.5}$$

At this point the RHS is simplified using Clebsch-Gordan coefficients.

A.2.1 Gaunt factors for j-j coupled initial and final states

Consider a transition between two fully J-resolved levels, we evaluate the $Q_{i,i'} R^I_{i,i'}(\nu,\nu')$ component of the Gaunt factor and split the position vector $\underline{r}$ into its radial and spherical harmonic components. The expression that must be evaluated becomes

$$\begin{aligned} \langle (S_p L_p J_p) nljJM|\underline{r}|(S'_p L'_p J'_p) n'l'j'J'M'\rangle &= \\ \langle (S_p L_p J_p) nljJM|rY_{1\mu}|(S'_p L'_p J'_p) n'l'j'J'M'\rangle \end{aligned} \tag{A.6}$$

Using the Wigner-Eckart thereom (equation (A.3)) to extract the dependence of the matrix on M and M'.

$$\begin{aligned} &\langle (S_p L_p J_p) nljJM|rY_{1\mu}|(S'_P L'_P J'_P) n'l'j'J'M'\rangle \\ = &\frac{C^{J'1J}_{M'\mu M}}{\sqrt{2J+1}} \langle (S_p L_p J_p) nljJ||rY_1||(S'_P L'_P J'_P) n'l'j'J'\rangle \end{aligned} \tag{A.7}$$

Multiply both sides by $\sum_{M'\mu} C^{J'1J'}_{M'\mu M}$ and use

$$\sum_{M'\mu} C^{J'1J}_{M'\mu M} C^{J'1J'}_{M'\mu M'} = \delta_{JJ'}\delta_{MM'} \tag{A.8}$$

to get

$$\begin{aligned} & \sum_{M'\mu} \langle (S_p L_p J_p) nljJM | rY_{1\mu} | (S'_p L'_p J'_p) n'l'j'J'M' \rangle C^{J'1J}_{M'\mu M} (2J+1)^{1/2} \\ = \; & \delta_{JJ'}\delta_{MM'} \langle (S_P L_P J_P) nljJ || rY_1 || (S'_P L'_P J'_P) n'l'j'J' \rangle \end{aligned} \tag{A.9}$$

Now sum both sides over M, note that the RHS in the above equation has no M dependence, and thus the sum over M introduces a factor $(2J+1)$ on the RHS.

$$\begin{aligned} & \langle (S_P L_P J_P) nljJ || rY_1 || (S'_P L'_P J'_P) n'l'j'J' \rangle \\ = \; & (2J+1)^{-1/2} \sum_{M'M\mu} C^{J'1J}_{M'\mu M} \langle (S_p L_p J_p) nljJM | rY_{1\mu} | (S'_p L'_p J'_p) n'l'j'J'M' \rangle \end{aligned} \tag{A.10}$$

Then use the Clebsch-Gordan coefficients (equation (A.1)) to reduce the matrix dependence to j, m, j' and m'.

$$\begin{aligned} LHS \; = \; & (2J+1)^{-1/2} \sum_{\substack{M'\mu \\ M_P m \\ M'_P m'}} C^{J'1J}_{M'\mu M} C^{J_p jJ}_{M_p m M} C^{J'_P j'J'}_{M'_P m' M'} \\ & \times \langle (S_p L_p J_p M_p) nljm | rY_{1\mu} | (S'_p L'_p J'_p M'_p) n'l'j'm' \rangle \end{aligned} \tag{A.11}$$

Since there is no change in the parent configuration this can be expressed as.

$$\begin{aligned} LHS \; = \; & (2J+1)^{-1/2} \delta(S_P L_P J_P M_P, S'_p L'_p J'_p M'_p) \sum_{\substack{M'\mu \\ M_P m \\ M'_P m'}} C^{J'1J}_{M'\mu M} C^{J_P jJ}_{M_p m M} C^{J_P j'J}_{M_p m' M'} \\ & \times \langle nljm | rY_{1\mu} | n'l'j'm' \rangle \\ = \; & (2J+1)^{-\frac{1}{2}} \delta(S_p L_p J_p, S'_p L'_p J'_p) \sum_{\substack{M'\mu \\ M_P m \\ m'}} C^{J'1J}_{M'\mu M} C^{J_p jJ}_{M_p m M} C^{J_p j'J'}_{M_p m' M'} \\ & \times < nl | r | n'l' >< ljm | Y_{1\mu} | l'j'm' > \end{aligned} \tag{A.12}$$

Then use equation (A.1) to uncouple j, m and j', m' into l, m_l, s, m_s and l', m_l', s', m_s'.

$$\begin{aligned} LHS &= (2J+1)^{-\frac{1}{2}}\delta(S_pL_pJ_p, S_p'L_p'J_p') \sum_{\substack{M'\mu M_p \\ mm'm_l \\ m_sm_l'm_s'}} C^{J'1J}_{M'\mu M} C^{J_pjJ}_{M_pmM} C^{J_pj'J'}_{M_pm'M'} C^{l\frac{1}{2}j}_{m_lm_sm} \\ &\times C^{l'\frac{1}{2}j'}_{m_l'm_s'm'} < nl|r|n'l' >< lm_lm_s|Y_{1\mu}|l'm_l'm_s' > \end{aligned} \quad (A.13)$$

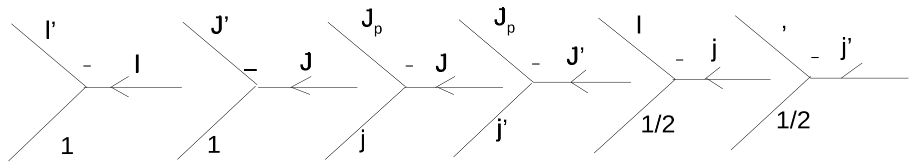
m_s must equal m_s' so

$$\begin{aligned} LHS &= (2J+1)^{-\frac{1}{2}}\delta(S_pL_pJ_pm_s, S_p'L_p'J_p'm_s') \sum_{\substack{M'\mu M_p \\ mm'm_l \\ m_sm_l'm_s'}} C^{J'1J}_{M'\mu M} C^{J_pjJ}_{M_pmM} C^{J_pj'J'}_{M_pm'M'} \\ &\times C^{l\frac{1}{2}j}_{m_lm_sm} C^{l'\frac{1}{2}j'}_{m_l'm_s'm'} < nl|r|n'l' >< lm_lm_s|Y_{1\mu}|l'm_l'm_s' > \end{aligned} \quad (A.14)$$

Using the Wigner-Eckart theorem once again we get

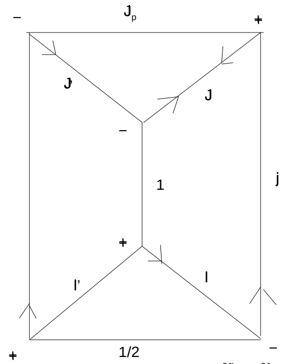
$$\begin{aligned} LHS &= (2J+1)^{-\frac{1}{2}}\delta(S_pL_pJ_p, S_p'L_p'J_p') \sum_{\substack{M'\mu M_p \\ mm'm_l \\ m_sm_l'}} C^{J'1J}_{M'\mu M} C^{J_pjJ}_{M_pmM} C^{J_pj'J'}_{M_pm'M'} \\ &\times C^{l\frac{1}{2}j}_{m_lm_sm} C^{l'\frac{1}{2}j'}_{m_l'm_sm'} \frac{C^{l'1l}_{m_l'\mu m_l}}{(2l+1)^{\frac{1}{2}}} < nl|r|n'l' >< l||Y_1||l' > \\ &= \frac{(2J+1)^{-\frac{1}{2}}}{(2l+1)^{\frac{1}{2}}}\delta(S_pL_pJ_p, S_p'L_p'J_p') < nl|r|n'l' >< l||Y_1||l' > \\ &\times \sum_{\substack{M'\mu M_p \\ mm'm_l \\ m_sm_l'}} C^{J'1J}_{M'\mu M} C^{J_pjJ}_{M_pmM} C^{J_pj'J'}_{M_pm'M'} C^{l\frac{1}{2}j}_{m_lm_sm} C^{l'\frac{1}{2}j'}_{m_l'm_sm'} C^{l'1l}_{m_l'\mu m_l} \end{aligned} \quad (A.15)$$

The Clebsch-Gordan coefficients are then represented graphically and equation (A.15) reduced to Wigner 6-j symbols as follows



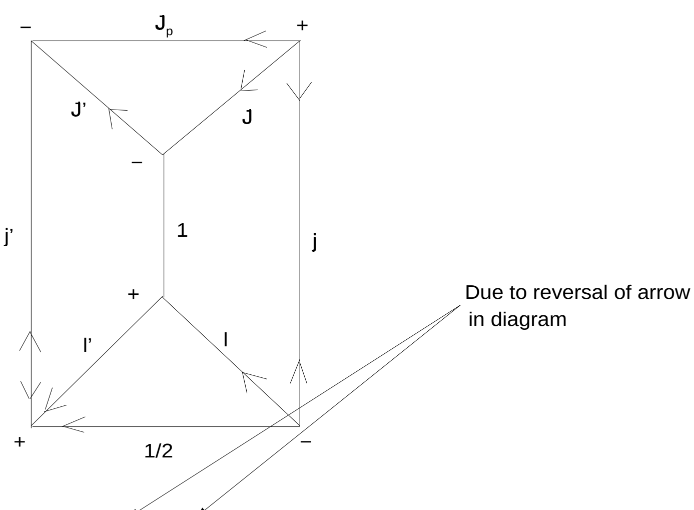
$(-1)^{2l'}(2l+1)^{1/2}$ $(-1)^{2J'}(2J+1)^{1/2}$ $(-1)^{2J_p}(2J+1)^{1/2}$ $(-1)^{2J_p}(2J'+1)^{1/2}$ $(-1)^{2l}(2j+1)^{1/2}$ $(-1)^{2l'}(2j'+1)^{1/2}$

≡



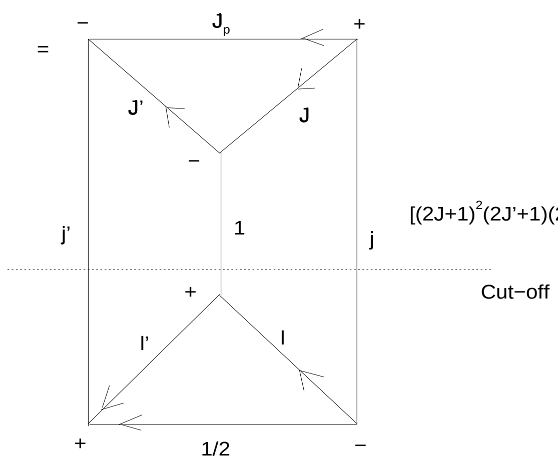
$(-1)^{2J'}(-1)^{2J_p}(-1)^{2J_p}(-1)^{2l}(-1)^{2l'}(-1)^{2l'}(2J+)^{1/2}(2J+1)^{1/2}(2J'+1)^{1/2}(2j+1)^{1/2}(2j'+1)^{1/2}(2l+1)^{1/2}$

=

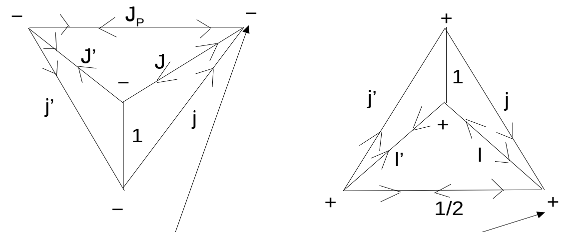


$(-1)^{2J'}(-1)^{2l}$ $(-1)^{2J'}(-1)^{2l}[(2J+1)^2(2J'+1)(2j+1)(2j'+1)(2l+1)]^{1/2}$

=



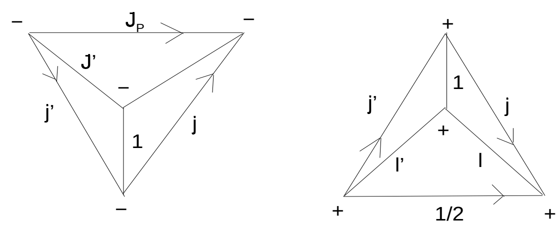
$[(2J+1)^2(2J'+1)(2l+1)(2j+1)(2j'+1)]^{1/2}$



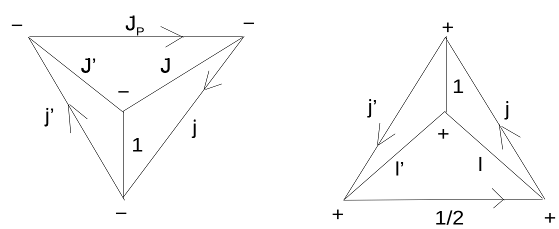
$[(2J+1)^2(2J'+1)(2l+1)(2j+1)(2j'+1)]^{1/2}$

$(-1)^{Jp+J+j}(-1)^{j+l+1/2}$

Note the change in sign at the node



$(-1)^{Jp+J+j}(-1)^{j+l+1/2}$ $[(2J+1)^2(2J'+1)(2l+1)(2j+1)(2j'+1)]^{1/2}$



$(-1)^{2j'} (-1)^{2Jp} (-1)^{Jp+J+j} (-1)^{2j'} (-1)^{2j} (-1)^{j+l+1/2}$

$[(2J+1)^2(2J'+1)(2l+1)(2j+1)(2j'+1)]^{1/2}$

$$= \left\{ \begin{array}{ccc} j & J & J_P \\ J' & j' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} j' & l' & 1/2 \\ l & j & 1 \end{array} \right\} (-1)^{2J_P}(-1)^{J_P+J}(-1)^{l+1/2}$$
$$\times[(2J+1)^2(2J'+1)(2j+1)(2j'+1)(2l+1)]^{1/2} \qquad \text{(A.16)}$$

Thus we have

$$LHS = \frac{(2J+1)^{-1/2}}{(2l+1)^{1/2}}\delta(S_PL_PJ_P;S'_PL'_PJ'_P) < nl|r|n'l' > |^2| < l||Y_1||l' > |^2$$
$$\times(-1)^{3J_P+J+l+1/2} \left\{ \begin{array}{ccc} j & J & J_P \\ J' & j' & 1 \end{array} \right\} \left\{ \begin{array}{ccc} j' & l' & 1/2 \\ l & j & 1 \end{array} \right\}$$
$$\times[(2J+1)^2(2J'+1)(2j+1)(2j'+1)(2l+1)]^{1/2} \qquad \text{(A.17)}$$

leading to the final result

$$|\langle(S_PL_PJ_P)nljJ||rY_1||(S'_PL'_PJ'_P)n'l'j'J'\rangle|^2$$
$$= (2J+1)(2j+1)(2j'+1)(2J'+1)\delta(S_PL_PJ_P;S'_PL'_PJ'_P)$$
$$\times| < nl|r|n'l' > |^2| < l||Y_1||l' > |^2 \left\{ \begin{array}{ccc} j & J & J_P \\ J' & j' & 1 \end{array} \right\}^2 \left\{ \begin{array}{ccc} j' & l' & 1/2 \\ l & j & 1 \end{array} \right\}^2 \text{(A.18)}$$

It is now possible to sum equation A.18 to account for all possible degrees of resolution in the transitions for the new coupling scheme. The results are shown in table A.1.

	$(S_PL_PJ_P)n'l'j'J'$	$(S_PL_PJ_P)n'l'j'$	$(S_PL_PJ_P)n'l'$	$(S_PL_PJ_P)n'$
$(S_pL_pJ_P)nljJ$	$(2J+1)(2J'+1)(2j+1)(2j'+1)$ $\left\{ \begin{matrix} j & J & J_P \\ J' & j' & 1 \end{matrix} \right\}^2$ $\left\{ \begin{matrix} j' & l' & 1/2 \\ l & j & 1 \end{matrix} \right\}^2$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$(2J+1)(2j'+1)$ $\left\{ \begin{matrix} j' & l' & 1/2 \\ l & j & 1 \end{matrix} \right\}^2$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2J+1)}{(2l+1)}$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2J+1)}{(2l+1)}$ $\sum_{l'}\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$
$(S_pL_pJ_P)nlj$	$(2J'+1)(2j+1)$ $\left\{ \begin{matrix} j' & l' & 1/2 \\ l & j & 1 \end{matrix} \right\}^2$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$(2j'+1)(2J_P+1)(2j+1)$ $\left\{ \begin{matrix} j' & l' & 1/2 \\ l & j & 1 \end{matrix} \right\}^2$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2J_P+1)(2j+1)}{(2l+1)}$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2J_P+1)(2j+1)}{(2l+1)}$ $\sum_{l'}\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$
$(S_pL_pJ_P)nl$	$\frac{(2J'+1)}{(2l'+1)}$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2j'+1)(2J_P+1)}{(2l'+1)}$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$2(2J_P+1)$ $\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$2(2J_P+1)$ $\sum_{l'}\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$
$(S_pL_pJ_P)n$	$\frac{(2J'+1)}{(2l'+1)}$ $\sum_l\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$\frac{(2j'+1)(2J_P+1)}{(2l'+1)}$ $\sum_l\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$(2J_P+1)$ $\sum_l\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$	$(2J_P+1)$ $\sum_{ll'}\vert <l\vert\vert Y_1\vert\vert l'>\vert^2\vert <nl\vert r\vert n'l'>\vert^2$

Table A.1: $(j-j)$ coupling to $(j'-j')$ coupling Q and R values evaluated for various levels of resolution.

A.2.2 Gaunt factors for cross-coupling

The angular algebra solution for the cross-coupling Gaunt factor proceeds as follow:

$$\begin{aligned}
& < (S_P L_P) nlSLJ || rY_1 || (S'_P L'_P J'_P) n'l'j'J' > \\
= & \ (2J+1)^{1/2} \sum_{M'\mu} C^{J'1J}_{M'\mu M} < (S_P L_P) nlSLJM | rY_{1\mu} | (S'_P L'_P J'_P) n'l'j'J'M' > \\
= & \ (2J+1)^{-1/2} \sum_{\substack{M'\mu M \\ M_S M_L M'_P \\ m'}} C^{J'1J}_{M'\mu M} C^{SLJ}_{M_S M_L M} C^{J'_P j' J'}_{M'_P m' M'} \\
& \times < (S_P L_P) nlSLM_L M_S | rY_{1\mu} | (S'_P L'_P J'_P M'_P) n'l'j'm' >
\end{aligned}$$

Note that in this last line the LHS is indepentent of M and therefore summing both sides over M introduces a $\frac{1}{(2J+1)}$ on the RHS, hence the change in sign of the $(2J+1)$ term.

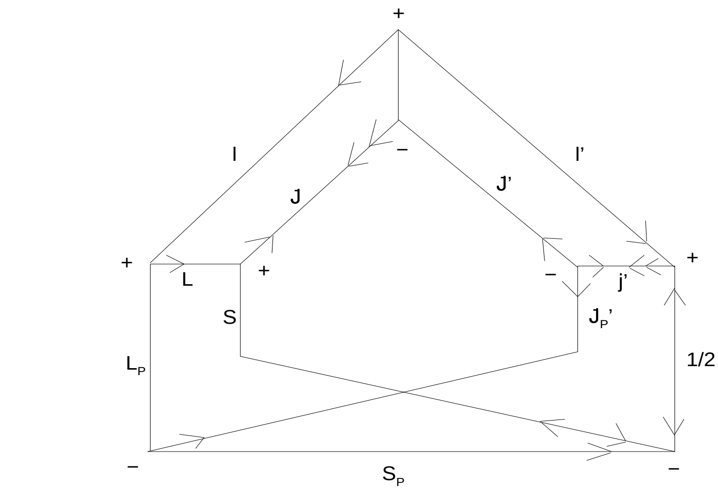
$$\begin{aligned}
LHS = & \ (2J+1)^{-1/2} \sum_{\substack{M'\mu M_S \\ M_L \\ M'_P m' \\ M_{S_P} m_s M_{L_P} \\ m_l m'_l m'_s \\ M'_{L_P} M'_{S_P}}} C^{J'1J}_{M'\mu M} C^{SLJ}_{M_S M_L M} C^{J'_P j' J'}_{M'_P m' M'} C^{S_P \frac{1}{2} S}_{M_{S_P} m_s M_S} C^{L_P l L}_{M_{L_P} m_l M_L} C^{l' \frac{1}{2} j'}_{m'_l m'_s m'} \\
& \times C^{L'_P S'_P J'_P}_{M_{L'_P} M_{S'_P} M_{P'}} < (S_P L_P M_{L_P} M_{S_P}) nlm_l m_s | rY_{1\mu} | (S'_P L'_P M_{L'_P} M_{S'_P}) n'l'm'_l m'_s > \\
= & \ (2J+1)^{-1/2} \delta(S_p L_p M_{L_P} M_{S_P} m_s; S'_p L'_p M'_{L_P} M'_{S_P} m'_s) < nlm_l | rY_{1\mu} | n'l'm'_s > \\
& \times \sum_{\substack{M'\mu M_S \\ M_L M'_P m' \\ M_{S_P} m_s M_{L_P} \\ m_l m'_l m'_s \\ M'_{L_P} M'_{S_P}}} C^{J'1J}_{M'\mu M} C^{SLJ}_{M_S M_L M} C^{J'_P j' J'}_{M'_P m' M'} \\
& \times C^{S_P \frac{1}{2} S}_{M_{S_P} m_s M_S} C^{L_P l L}_{M_{L_P} m_l M_L} C^{l' \frac{1}{2} j'}_{m'_l m'_s m'} C^{L'_P S'_P J'_P}_{M_{L'_P} M_{S'_P} M_{P'}}
\end{aligned}$$

$$= \frac{(2J+1)^{-1/2}}{(2l+1)^{1/2}} \sum_{\substack{M'\mu M_S \\ M_L M'_P m' \\ M'_{S_P} m_s M_{L_P} \\ m_l m'_l}} C^{J'1J}_{M'\mu M} C^{SLJ}_{M_S M_L M} C^{J'_P j' J'}_{M'_P m' M'}$$

$$\times C^{S_P \frac{1}{2} S}_{M_{S_P} m_s M_S} C^{L_P l L}_{M_{L_P} m_l M_L} C^{l' \frac{1}{2} j'}_{m'_l m_s m'} C^{L_P S_P J'_P}_{M_{L_P} M_{S_P} M_{P'}} C^{l' 1 l}_{M_{l'} \mu M_l}$$

$$\times < nl|r|n'l' >< l||Y_1||l' > \tag{A.19}$$

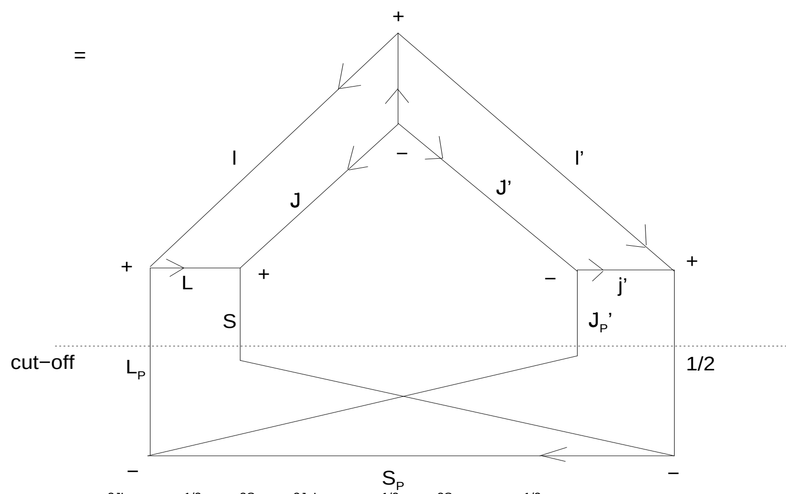
One can then use the graphical methods as before to simplify the Clebsch-Gordan coefficients.



$$(2J+1)^{-1/2}(2l+1)^{1/2}(-1)^{2J'}(2J+1)^{1/2}(-1)^{2S}(2J+1)^{1/2}(-1)^{2Jp'}(2J'+1)^{1/2}$$

$$(-1)^{2Sp}(2S+1)^{1/2}(-1)^{2Lp}(2L+1)^{1/2}(-1)^{2l'}(2j'+1)^{1/2}(-2)^{2Lp}(2J_p'+1)^{1/2}(-1)^{2l'}(2l+1)^{1/2}$$

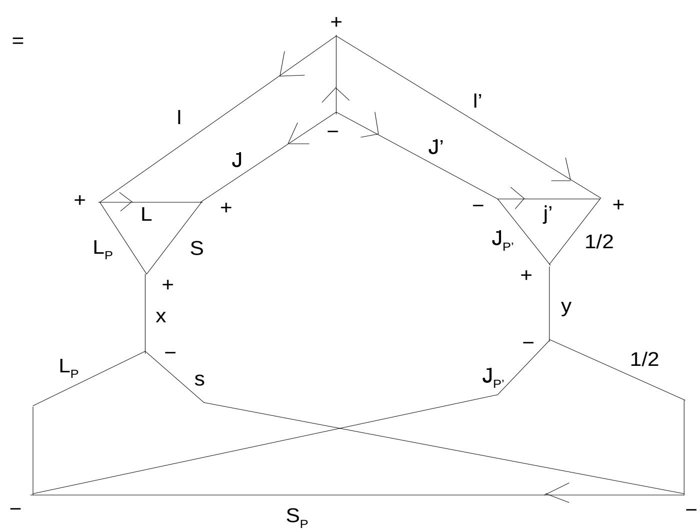
=



$$(-1)^{2J'}(2J+1)^{1/2}(-1)^{2S}(-1)^{2Jp'}(2J'+1)^{1/2}(-1)^{2Sp}(2S+1)^{1/2}$$

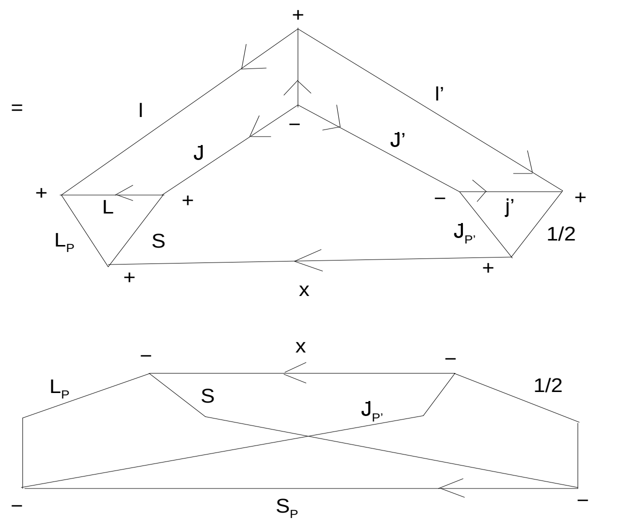
$$(2L+1)^{1/2}(2j'+1)^{1/2}(2J_P'+1)^{1/2}(-1)^{2J'}(-1)^{2Sp}$$

=

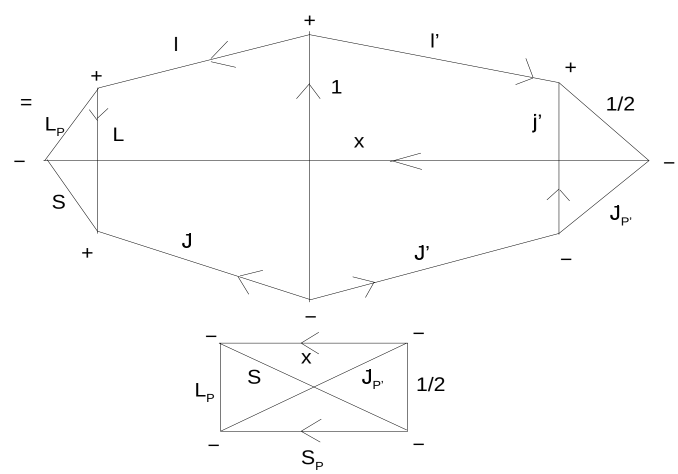


$$\Sigma(2x+1)(2y+1)$$

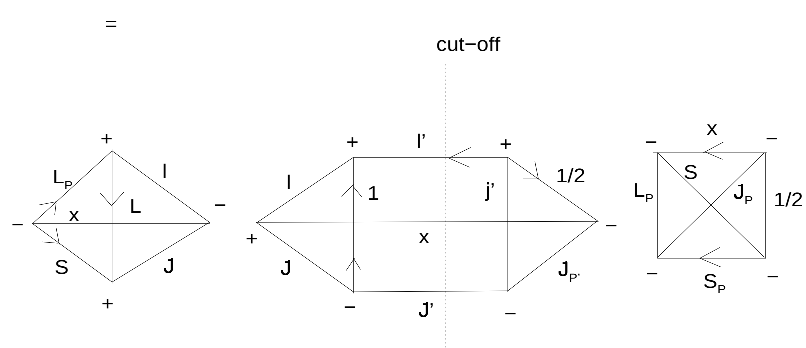
$$(-1)^{2S}(-1)^{2Jp'}(2J+1)^{1/2}(2J'+1)^{1/2}(2S+1)^{1/2}(2L+1)^{1/2}(2j'+1)^{1/2}(2J_{P'}+1)^{1/2}$$



$$\Sigma(2x+1)\quad (-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{P'}+1)]^{1/2}$$

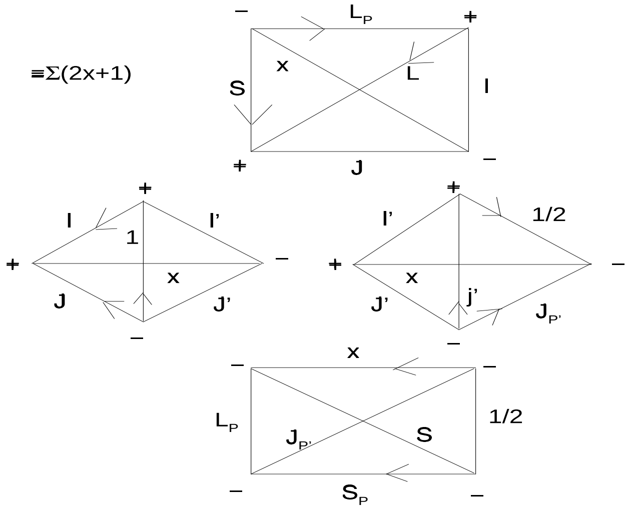


$$\Sigma(2x+1)(-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{P'}+1)]^{1/2}$$



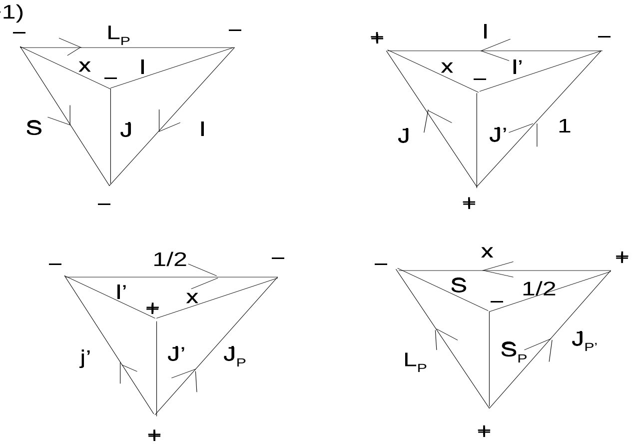
$$\Sigma(2x+1)\quad (-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_P'+1)]^{1/2}$$

$\equiv\Sigma(2x+1)$



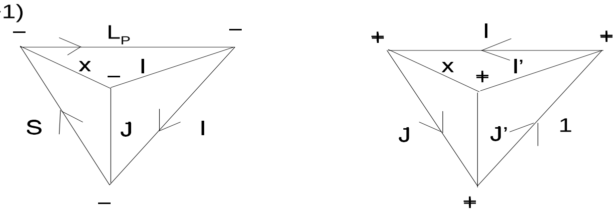
$$(-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{P'}+1)]^{1/2}$$

$\equiv\Sigma(2x+1)$

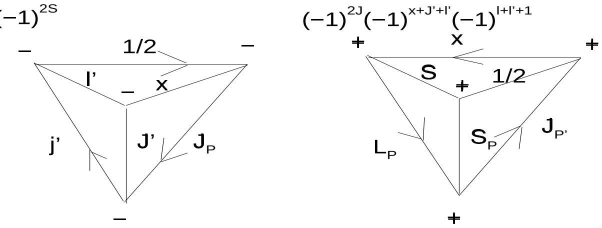


$$(-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{p'}+1)]^{1/2}$$

$\equiv\Sigma(2x+1)$



$(-1)^{2S}$ $\quad$ $(-1)^{2J}(-1)^{x+J'+l'}(-1)^{l+l'+1}$



$(-1)^{2Jp'}(-1)^{l'+J'+x}(-1)^{j'+J'+Jp'}$ $\quad$ $(-1)^{2Lp}(-1)^{1/2+S+Sp}(-1)^{x+Lp+S}$

$$(-1)^{2S}(-1)^{2Jp'}[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{p'}+1)]^{1/2}$$

$$
\begin{aligned}
= \; & \Sigma(2x+1) \left\{ \begin{array}{ccc} L & l & L_P \\ x & S & J \end{array} \right\} \left\{ \begin{array}{ccc} 1 & l' & l \\ x & J & J' \end{array} \right\} \left\{ \begin{array}{ccc} J'_P & x & 1/2 \\ l' & j' & J' \end{array} \right\} \left\{ \begin{array}{ccc} J'_P & 1/2 & x \\ S & L_P & S_P \end{array} \right\} \\
& \times(-1)^{2J}(-1)^{3x}(-1)^{3J'}(-1)^{3l'}(-1)^{l}(-1)^{\frac{3}{2}}(-1)^{j'}(-1)^{J'_P}(-1)^{3L_P}(-1)^{2S}(-1)^{S_P} \\
& \times[(2J+1)(2J'+1)(2S+1)(2L+1)(2j'+1)(2J_{P'}+1)]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
= \; & (-1)^{2J+3J'+3l'+l+3/2+j'+J'_P+3L_P+2S+S_P} \\
& \times(2J+1)^{1/2}(2J'+1)^{1/2}(2S+1)^{1/2}(2L+1)^{1/2}(2j'+1)(2J'_P+1)^{1/2} \\
& \times\Sigma_x(2x+1)(-1)^{3x} \left\{ \begin{array}{ccc} L & l & L_P \\ x & S & J \end{array} \right\} \left\{ \begin{array}{ccc} 1 & l' & l \\ x & J & J' \end{array} \right\} \\
& \times \left\{ \begin{array}{ccc} J'_P & x & 1/2 \\ l' & j' & J' \end{array} \right\} \left\{ \begin{array}{ccc} J'_P & 1/2 & x \\ S & L_P & S_P \end{array} \right\} \qquad \text{(A.20)}
\end{aligned}
$$

The results for the various levels of resolution can then be evaluated as before.